Optimal RH_2- and RH_∞-Approximation of Unstable Descriptor Systems

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Stability perserving is an important topic in approximation of systems, e.g. model reduction. If the original system is stable, we often want the approximation to be stable. But even if an algorithm preserves stability the resulting system could be unstable in practice because of round-off errors. Our approach is approximating this unstable reduced system by a stable system. More precisely, we consider the following problem. Given an unstable linear time-invariant continuous-time descriptor system with transfer function G, find a stable one whose transfer function is the best approximation of G in the spaces RH_2 and RH_∞, respectively. Explicit optimal solutions are presented under consideration of numerical issues.

1 Introduction

We deal with linear time-invariant descriptor systems described by differential algebraic equations (see e.g. [KM06], [Dai89] for more details)

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
\tag{1.1}
$$

with matrices $(E, A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ and the corresponding transfer function $s \mapsto C(sE - A)^{-1}B + D$. In many applications such as model reduction (see [ASG01] for an overview an further references) or system identification (see [Lju99]) one wishes to approximate systems such that the transfer functions of the original and the approximated system are as close as possible. An important property of an approximation technique is preservation of stability (cf. e.g. balanced truncation [ASG01], hankel norm approximation [Glo84]). However, not every method has this property in general (Arnoldi method, Lanczos method [ASG01]). Moreover, a stable approximation can be unstable in practice [BFF98]. This happens in computer implementations because of round-off errors. One way out is restarting the algorithm with

changed parameters, e.g. interpolation points in Lanczos method. The disadvantage is at least doubled computational complexity. Another approach is to modify the algorithm directly (e.g. partial Padé-via-Lanczos method [BF01]).

In this paper our approach for stability perserving is to consider the computed unstable approximation of the original system. Given is an unstable descriptor system S (1.1) with transfer function G. Find a stable descriptor system whose transfer function is the best approximation of G in the spaces RL_2 and RL_∞ of real rational functions on the imaginary axis $i\mathbb{R}$ in the Lebesgue spaces L_2 and L_∞, respectively. We call this system an optimal RH_2-approximation and optimal RH_∞-approximation of S, respectively.

For special cases this problem was already solved. The characterization of the unique optimal RH_2-approximation of a standard system (i.e. $E = I$ in (1.1)) with $D = 0$ is a simple consequence of the Paley-Wiener theorem. The set of all suboptimal RH_∞-approximations of standard systems was determined in [GGLD90] with j-spectral factorizations to solve model matching problems. However, no optimal ones were explicitly given. An explicit representation of a nonunique optimal RH_∞-approximation of minimal antistable standard systems was presented in the proof of [Glo84, Theorem 6.1]. This theorem was only used as an auxiliary result for a more general problem, the optimal hankel norm approximation. That representation requires the possibly ill-conditioned balanced minimal realization of G. This numerical issue was discussed in [SCL90].

Our approach for approximating an unstable descriptor system S in a numerically reliable way is the following. First, we decompose S into a stable and antistable part by combining [GL96, Theorem 7.7.2] and [KD92, Section 4.1]. We show that it is sufficient to replace the antistable part by its approximation to get an approximation of S. This was e.g. proposed in [Fra87, Section 8.3] for standard systems and w.r.t. RL_∞. The stable part is the unique optimal RH_2-approximation of S. Our representation of the optimal RH_∞-approximation only requires at most one singular value decomposition. For discret-time standard systems [Mar00] presented an approximation method similar to ours. However, in contrast to our approach they used a balanced minimal realization of the antistable part.

The paper is organized as follows. In Section 2 we summarize some results regarding the theory of matrices and matrix pairs. We introduce the setting and recall some system transformations, e.g. balanced realization. Section 3 states and solves both approximation problems. Section 4 presents an algorithm solving our problem and some numerical examples.

2 Preliminaries

We define $\mathbb{C}_{>0} := \{s \in \mathbb{C};\ \mathrm{Re}(s) > 0\}$ and analogously $\mathbb{C}_{\geq 0}$, $\mathbb{C}_{<0}$, $\mathbb{C}_{\leq 0}$. The set $i\mathbb{R}$ is the imaginary axis. For a function f we denote its domain by $\mathcal{D}(f)$ and for $M \subseteq \mathcal{D}(f)$ the image of M under f by $f[M]$. The kernel of a matrix $A \in \mathbb{C}^{n \times m}$ is denoted by $\ker(A)$, the adjoint by A^*, the spectrum by $\sigma(A)$ and the Frobenius norm by $\|A\|_F = \sqrt{\mathrm{trace}(A^*A)}$.

The matrix $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix and $0_{n \times m} \in \mathbb{R}^{n \times m}$ the zero matrix. The vector $e_i \in \mathbb{R}^n$ is the ith unit vector.

Definition 2.1. We call $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ a *matrix pair*. A scalar $\lambda \in \mathbb{C}$ is called a *(generalized) eigenvalue* of (E, A), if $\det(\lambda E - A) = 0$. If there exists $v \neq 0$ such that $Ev = 0$ and $Av \neq 0$, then ∞ is called a (generalized) eigenvalue of (E, A). We denote the set of all eigenvalues of (E, A) by $\sigma(E, A) \subseteq \mathbb{C} \cup \{\infty\}$ and the resolvent set by $\rho(E, A) := \mathbb{C} \backslash \sigma(E, A)$. If $\rho(E, A) = \emptyset$, then (E, A) is called *singular* otherwise *regular*. We denote the set of all regular matrix pairs by $\mathbb{P}_n := \{(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}; \ (E, A) \ \text{regular}\}$.

Remark 2.2. If the matrix pair is regular, then the set of eigenvalues is finite. Otherwise every $\lambda \in \mathbb{C}$ is an eigenvalue. Details about the generalized eigenvalue problem can be found in [Ste72].

To prepare decompositions of systems, we combine the results in [GL96, Theorem 7.7.2] and [KD92, Section 4.1] to get the following theorem.

Theorem 2.3. *Let $(E, A) \in \mathbb{P}_n$ and $M_1, M_2 \subseteq \mathbb{C} \cup \{\infty\}$ be two disjoint sets such that $\sigma(E, A) = M_1 \cup M_2$. Then there exist orthogonal $U, V \in \mathbb{R}^{n \times n}$ such that*

$$(UEV, UAV) = \left(\begin{pmatrix} E_1 & E_2 \\ 0 & E_3 \end{pmatrix}, \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \right), \tag{2.1}$$

$\sigma(E_1, A_1) = M_1$ and $\sigma(E_3, A_3) = M_2$. Moreover, there exist $R, L \in \mathbb{R}^{m \times p}$ such that

$$\begin{aligned} A_1 R - L A_3 &= -A_2 \\ E_1 R - L E_3 &= -E_2 \, . \end{aligned} \tag{2.2}$$

The matrices $P := \begin{pmatrix} I & -L \\ 0 & I \end{pmatrix} U, \ Q := V \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}$ satisfy

$$(PEQ, PAQ) = \left(\begin{pmatrix} E_1 & 0 \\ 0 & E_3 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix} \right).$$

Remark 2.4. We get (2.1) e.g. by applying the generalized Schur decomposition to (E, A). Equation (2.2) is called generalized (or coupled) Sylvester equation (see [KD92]).

We recall the following known result which can be used to test regularity of matrices.

Theorem 2.5 ([ZDG96, Section 2.3]). *Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}^{m \times m}$ and $M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. If A is regular, then we have $\det(M) = \det(D - CA^{-1}B)\det(A)$. If D is regular, then we have $\det(M) = \det(A - BD^{-1}C)\det(D)$. The matrices $D - CA^{-1}B$ and $A - BD^{-1}C$ are called the* Schur complement *of A and D in M, respectively.*

Definition 2.6. For $n, p, m \in \mathbb{N}$ we define the sets of *systems* (compare (1.1))

$$\begin{aligned} \mathbb{S}_{n,p,m} &:= \mathbb{P}_n \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}, \\ \mathbb{S}_{n,p,m}^0 &:= \{(E, A, B, C, D) \in \mathbb{S}_{n,p,m}; \ i\mathbb{R} \subseteq \rho(E, A)\}, \\ \mathbb{S}_{n,p,m}^+ &:= \{(E, A, B, C, D) \in \mathbb{S}_{n,p,m}; \ \mathbb{C}_{\geq 0} \subseteq \rho(E, A)\}, \\ \mathbb{S}_{n,p,m}^- &:= \{(E, A, B, C, D) \in \mathbb{S}_{n,p,m}; \ \mathbb{C}_{\leq 0} \subseteq \rho(E, A), \ E \ \text{regular}\}, \end{aligned}$$

where $\mathbb{S}_{n,p,m}^{+}$ and $\mathbb{S}_{n,p,m}^{-}$ are the set of the *stable* and *antistable* systems, respectively. We call a system $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}$ a *standard system* if $E = I$ and a *descriptor system* otherwise. For $S_i = (E_i, A_i, B_i, C_i, D_i) \in \mathbb{S}_{n_i,p,m}$, $i \in \{1, 2\}$, and regular $P, Q \in \mathbb{R}^{n_1 \times n_1}$ we define the operations

$$S_1 \oplus S_2 := \left(\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D_1 + D_2 \right),$$

$$P \cdot S_1 \cdot Q := (PE_1Q, PA_1Q, PB_1Q, C_1Q, D_1),$$

$$P \cdot S_1 := P \cdot S_1 \cdot I_{n_1}, \qquad\qquad S_1 \cdot Q := I_{n_1} \cdot S_1 \cdot Q.$$

Some of the following definitions and theorems for standard systems can be given for descriptor systems, too. However, we restricted ourselves to the specializations we need here.

Definition 2.7 ([Dai89, Definition 2-6.1]). Let $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}$. Then

$$\mathcal{G}(S) \colon \rho(E, A) \ni s \longmapsto C(sE - A)^{-1}B + D \in \mathbb{R}^{p \times m}$$

is called the *transfer function* of S. We denote the set of all $p \times m$ transfer functions by $\mathbb{T}_{p,m} := \{\mathcal{G}(S); \; S \in \bigcup_{n \in \mathbb{N}} \mathbb{S}_{n,p,m}\}$. A system $\widetilde{S} \in \mathbb{S}_{\widetilde{n},p,m}$ is called a *realization* of $\mathcal{G}(S)$, if $\mathcal{G}(S)(s) = \mathcal{G}(\widetilde{S})(s)$ for all $s \in \mathcal{D}(\mathcal{G}(S)) \cap \mathcal{D}(\mathcal{G}(\widetilde{S}))$. We denote the set of all realizations of $\mathcal{G}(S)$ by $\mathbb{S}(\mathcal{G}(S))$. A realization $S_1 \in \mathbb{S}_{n_1,p,m}$ of $\mathcal{G}(S)$ is said to be *minimal*, if every other realization $S_2 \in \mathbb{S}_{n_2,p,m}$ of $\mathcal{G}(S)$ fulfils $n_2 \geq n_1$.

Remark 2.8. Transfer functions are meromorphic. Obviously, $\mathcal{G}(S_1 \oplus S_2) = \mathcal{G}(S_1) + \mathcal{G}(S_2)$ and $\mathcal{G}(P \cdot S \cdot Q) = \mathcal{G}(S)$ hold.

We want to transform a system $S = (E, A, B, C, D)$ into $\widetilde{S} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ with $\mathcal{G}(S) = \mathcal{G}(\widetilde{S})$, in particular $\sigma(E, A) = \sigma(\widetilde{E}, \widetilde{A})$, such that $\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ have a special structure, e.g. to decompose systems into a stable and antistable part. Note that all matrices stay real under the presented transformations.

Definition 2.9 ([Dai89, Definition 1-3.1]). Two systems $S_i = (E_i, A_i, B_i, C_i, D_i) \in \mathbb{S}_{n,p,m}$, $i \in \{1, 2\}$, are called *restricted system equivalent* (for short r.s.e., $S_1 \sim S_2$), if there exist regular $P, Q \in \mathbb{R}^{n \times n}$ such that $P \cdot S_1 \cdot Q = S_2$.

Lemma 2.10. *For every $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}$ there exist $S_w \sim S$, $k \in \{0, \ldots, n\}$ and $\nu \in \mathbb{N}$ such that*

$$S_w = \left(\begin{pmatrix} I_k & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} J & 0 \\ 0 & I_{n-k} \end{pmatrix}, \begin{pmatrix} B_J \\ B_N \end{pmatrix}, \begin{pmatrix} C_J & C_N \end{pmatrix}, D \right),$$

and $N^\nu = 0$. Moreover, $\mathcal{D}(\mathcal{G}(S)) = \rho(J) = \rho(E, A)$ holds and

$$\forall s \in \mathcal{D}(\mathcal{G}(S)) \colon \mathcal{G}(S)(s) = C_J(sI - J)^{-1}B_J + D - \sum_{i=0}^{\nu-1} s^i C_N N^i B_N. \qquad (2.3)$$

Proof. We apply Theorem 2.3 to (E, A) with $M_1 = \sigma(E, A)\backslash\{\infty\}$, $M_2 = \{\infty\}$ and get regular E_1, A_3, P, Q as in the theorem. Choose $\widetilde{P} := \begin{pmatrix} E_1^{-1} & 0 \\ 0 & I \end{pmatrix} P$, $\widetilde{Q} := \begin{pmatrix} I & 0 \\ 0 & A_3^{-1} \end{pmatrix} Q$ and set $S_w := \widetilde{P} \cdot S \cdot \widetilde{Q}$. The matrix $N := E_3$ is nilpotent because $\sigma(N, I) = \{\infty\}$ and thus $\sigma(N) = \{0\}$. Equation (2.3) follows from $\mathcal{G}(S) = \mathcal{G}(S_w)$ (Remark 2.8) and $C_N(sN - I)^{-1} B_N = -\sum_{i=0}^{\nu-1} s^i C_N N^i B_N$ (Neumann series). $\qquad \square$

Theorem 2.11 ([ZDG96, Theorem 3.10, Theorem 3.17]). *Let $S = (I, A, B, C, D) \in \mathbb{S}_{n,p,m}$. Then*

$$S \sim \left(I_n, \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & C_3 & 0 \end{pmatrix}, D \right),$$

where $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $n_i \in \{0, \ldots, n\}$ for $i \in \{1, 2, 3, 4\}$ and $S_M := (I, A_{11}, B_1, C_1, D)$ is a minimal realization of $\mathcal{G}(S)$. This decomposition is known as the Kalman decomposition. All minimal realizations of $\mathcal{G}(S)$ are r.s.e. to S_M.

Now we introduce the controllability and observability Gramians. Later these matrices will play a central role in our problem of computing an optimal RH_∞-approximation. The next theorem follows easily from [Pen98, Corollary 2].

Theorem 2.12 ([Pen98], Lyapunov equation). *Let $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$. Then there exist $X_c, X_o \in \mathbb{R}^{n \times n}$ such that*

$$\begin{aligned} A X_c E^\top + E X_c A^\top + B B^\top &= 0, \\ A^\top X_o E + E^\top X_o A + C^\top C &= 0. \end{aligned} \tag{2.4}$$

They are unique and symmetric. We denote them by $\mathfrak{C}_S := X_c$ (controllability Gramian of system S) and $\mathfrak{O}_S := X_o$ (observability Gramian).

Lemma 2.13. *Let $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}^+$ and E be regular. Let $P, Q \in \mathbb{R}^{n \times n}$ be regular. Then $\mathfrak{C}_S = Q \mathfrak{C}_{P \cdot S \cdot Q} Q^\top$ and $\mathfrak{O}_S = P^\top \mathfrak{O}_{P \cdot S \cdot Q} P$ hold.*

Proof. We prove this only for $\mathfrak{C}_S$ since $\mathfrak{O}_S$ is analogously. By definition of $\mathfrak{C}_{P \cdot S \cdot Q}$

$$\begin{aligned} 0 &= PAQ\mathfrak{C}_{P \cdot S \cdot Q} Q^\top E^\top P^\top + PEQ\mathfrak{C}_{P \cdot S \cdot Q} Q^\top A^\top P^\top + PBB^\top P^\top, \\ &= P\left(A(Q\mathfrak{C}_{P \cdot S \cdot Q} Q^\top) E^\top + E(Q\mathfrak{C}_{P \cdot S \cdot Q} Q^\top) A^\top + BB^\top \right) P^\top \end{aligned}$$

hold. Due to the regularity of P and Q and the uniqueness of the solution of (2.4) (Theorem 2.12) the assertion is proved. $\qquad \square$

Theorem 2.14 ([Glo84, Theorem 4.3]). *Let* $S = (I, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$. *Then there exist* $S_b \sim S$, $r \in \mathbb{N}$, $h \in \mathbb{R}_{\geq 0}$ *such that*

$$\mathfrak{C}_{S_b} = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -hI_r \end{pmatrix}, \qquad \mathfrak{D}_{S_b} = \begin{pmatrix} \Sigma_2 & 0 \\ 0 & -hI_r \end{pmatrix}, \qquad h^2 \notin \sigma(\Sigma_1 \Sigma_2).$$

If $\Sigma_1 = \Sigma_2$ *are diagonal, then* S_b *is known as the* balanced realization *of* $\mathcal{G}(S)$.

To check whether $\lambda \in \mathbb{C}$ is an eigenvalue of (E, A), we need the following easy generalization of a known result for standard systems (e.g. [Mac04, Theorem 5.6.1 c]).

Lemma 2.15. *Let* $G \in \mathbb{T}_{p,m}$ *have a realization* $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}$. *Then for every finite* $\lambda \in \sigma(E, A)$ *at least one of the following three statements holds:*

(i) λ *is a pole of* G, $\qquad (ii)$ $\operatorname{rank}\begin{pmatrix} \lambda E - A & B \end{pmatrix} < n$, $\qquad (iii)$ $\operatorname{rank}\begin{pmatrix} \lambda E - A \\ C \end{pmatrix} < n$.

Proof. Let $S_w \sim S$ be as in Lemma 2.10. We set $S_J := (I, J, B_J, C_J, D)(\in \mathbb{S}_{k,p,m})$. By Lemma 2.10 we know that $\lambda \in \sigma(E, A) \setminus \{\infty\} \Leftrightarrow \lambda \in \sigma(J)$. By [Mac04, Theorem 5.6.1 c] one of the following three statements holds:

(i') λ *is a pole of* $\mathcal{G}(S_J)$, $\quad (ii')$ $\operatorname{rank}\begin{pmatrix} \lambda I - J & B_J \end{pmatrix} < k$, $\quad (iii')$ $\operatorname{rank}\begin{pmatrix} \lambda I - J \\ C_J \end{pmatrix} < k$.

Equation (2.3) shows that $(i) \Leftrightarrow (i')$. By the regularity of $\lambda N - I$ we have

$$\operatorname{rank}\begin{pmatrix} \lambda I - J & B_J \end{pmatrix} + (n - k) = \operatorname{rank}\begin{pmatrix} \lambda I - J & 0 & B_J \\ 0 & \lambda N - I & B_N \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \lambda E - A & B \end{pmatrix}$$

and analogously $\operatorname{rank}\begin{pmatrix} \lambda I - J \\ C_J \end{pmatrix} + (n - k) = \operatorname{rank}\begin{pmatrix} \lambda E - A \\ C \end{pmatrix}$. Thus $(ii) \Leftrightarrow (ii')$ and $(iii) \Leftrightarrow (iii')$. This finishes the proof. $\qquad\square$

Definition 2.16 ([ZDG96, Section 4.3]). We approximate unstable systems with respect to the following spaces and corresponding norms. Let $p, m \in \mathbb{N}$.

- $RL_\infty^{p \times m} := \{G \in \mathbb{T}_{p,m}; \ \mathrm{i}\mathbb{R} \subseteq \mathcal{D}(G), \ \|G\|_\infty < \infty\}$, $\|G\|_\infty := \sup\limits_{\omega \in \mathbb{R}} \|G(\mathrm{i}\omega)\|_2$;

- $RH_\infty^{p \times m} := \left\{G \in RL_\infty^{p \times m}; \ G \text{ analytic on } \mathbb{C}_{>0}\right\}$;

- $RL_2^{p \times m} := \{G \in \mathbb{T}_{p,m}; \ \mathrm{i}\mathbb{R} \subseteq \mathcal{D}(G), \ \|G\|_2 < \infty\}$, $\|G\|_2 := \sqrt{\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \|G(\mathrm{i}\omega)\|_F^2 \, d\omega}$;

- $RH_2^{p \times m} := \left\{G \in RL_2^{p \times m}; \ G \text{ analytic on } \mathbb{C}_{>0}\right\}$.

We omit the indices, if there is no risk of confusion.

Remark 2.17. These spaces are often introduced as a subset of socalled real rational functions, i.e. entrywise a quotient of polynomials with real coefficients. Actually, every real rational function is a transfer function (see [Dai89, Theorem 2-6.3]) and vice versa, since $G(s)_{i,j} = e_i^\top C(sE - A)^{-1} Be_j + D_{i,j}$ is the Schur complement of $sE - A$ in $M(s) := \begin{pmatrix} sE-A & Be_j \\ e_i^\top C & D_{i,j} \end{pmatrix}$ and thus $G(s)_{i,j} = \det(G(s)_{i,j}) = \frac{\det(M(s))}{\det(sE-A)}$ by Theorem 2.5. Let F and G be elements of one of these spaces with $\mathbb{S}(F) = \mathbb{S}(G)$, e.g. $s \mapsto \frac{1}{s-1}$ and $s \mapsto \frac{s-2}{(s-1)(s-2)}$. By definition they are equal on the cofinite set $\mathcal{D}(F) \cap \mathcal{D}(G)$. Therefore we identify them as the same function. With this and by the identity theorem for analytic functions the spaces are indeed normed spaces.

Theorem 2.18 ([Fra87, Section 2.3])**.** *Let $G_- \in RL_2$ be analytic on $\mathbb{C}_{<0}$ and $G_+ \in RH_2$. Then $\|G_- + G_+\|_2^2 = \|G_-\|_2^2 + \|G_+\|_2^2$ holds.*

The following known result characterizes these spaces in terms of realizations of transfer functions.

Theorem 2.19 ([Fra87, Section 2.3])**.** *Let $G \in \mathbb{T}_{p,m}$ with $i\mathbb{R} \subseteq \mathcal{D}(G)$. Then the following hold:*

(a) $G \in RL_\infty \Leftrightarrow \exists\,(I, A, B, C, D) \in \mathbb{S}(G)$,

(b) $G \in RH_\infty \Leftrightarrow \exists\,(I, A, B, C, D) \in \mathbb{S}(G) \cap \mathbb{S}^+_{n,p,m}$,

(c) $G \in RL_2 \Leftrightarrow \exists\,(I, A, B, C, 0) \in \mathbb{S}(G)$,

(d) $G \in RH_2 \Leftrightarrow \exists\,(I, A, B, C, 0) \in \mathbb{S}(G) \cap \mathbb{S}^+_{n,p,m}$.

3 Optimal stable approximations

3.1 Problem statement

In this paper, we deal with the following problem.

Approximation-Problem (AP$_q$). *Let $q \in \{2, \infty\}$. For $S \in \mathbb{S}^0_{n,p,m}$ find $\widetilde{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}^+_{\widehat{n},p,m}$ such that*

$$\|\mathcal{G}(S) - \mathcal{G}(\widetilde{S})\|_q = \inf_{\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}^+_{\widehat{n},p,m}} \|\mathcal{G}(S) - \mathcal{G}(\widehat{S})\|_q\,.$$

In the next subsections we show that this problem is solvable. We also present an explicit solution in Theorem 3.2 for $q = 2$ and Theorem 3.16 for $q = \infty$.

We do not consider systems $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}$ where (E, A) has imaginary eigenvalues. If there is no imaginary pole of $\mathcal{G}(S)$, then there exists a realization $S_2 \in \mathbb{S}^0_{n,p,m}$ of $\mathcal{G}(S)$ and we solve (AP$_q$) for S_2. Otherwise, for every $\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}^+_{\widehat{n},p,m}$ we have $\|\mathcal{G}(S) - \mathcal{G}(\widehat{S})\|_q = \infty$ and hence (AP$_q$) is not solvable.

First, we show that solving (AP_q) for a descriptor system we can equivalently solve (AP_q) for its antistable part. This was e.g. proposed in [Fra87, Section 8.3] but only for standard systems and $q = \infty$. We need Theorem 3.1 to use some results in [Glo84] and [SCL90] which are only applicable for antistable standard systems.

Theorem 3.1. *Let $S \in \mathbb{S}_{n,p,m}^0$. Then there exist $S_+ = (E_+, A_+, B_+, C_+, D) \in \mathbb{S}_{n_+,p,m}^+$ and $S_- = (E_-, A_-, B_-, C_-, 0) \in \mathbb{S}_{n_-,p,m}^-$ such that $S \sim S_+ \oplus S_-$. Let $q \in \{2, \infty\}$ and $\gamma \geq 0$. Then the following two are equivalent:*

$(i)\ \exists\, \widetilde{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+ \colon \|\mathcal{G}(S) - \mathcal{G}(\widetilde{S})\|_q \leq \gamma\,, \quad (ii)\ \exists\, \widetilde{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+ \colon \|\mathcal{G}(S_-) - \mathcal{G}(\widetilde{S})\|_q \leq \gamma\,.$

If $\widetilde{S}$ satisfies (ii), then $\widetilde{S} \oplus S_+$ satisfies (i).

Proof. Theorem 2.3 assures the existence of such systems S_+ and S_-. Let $\widetilde{S} \in \mathbb{S}_{\widehat{n},p,m}^+$ satisfy (i). Then we have

$$\gamma \geq \|\mathcal{G}(S) - \mathcal{G}(\widetilde{S})\|_q = \|\mathcal{G}(S_-) - (\mathcal{G}(\widetilde{S}) - \mathcal{G}(S_+))\|_q\,.$$

Thus $\widehat{S} := \widetilde{S} \oplus (E_+, A_+, B_+, -C_+, -D)$ satisfies (ii) (compare Remark 2.8). The fact that $\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+$ follows easily from the definition of $\oplus$. Let $\widetilde{S} \in \mathbb{S}_{\widehat{n},p,m}^+$ fulfils (ii). Then analogously $\widetilde{S} \oplus S_+$ satisfies (i). $\qquad\square$

3.2 Optimal RH_2-approximations

Theorem 3.2. *Let $S \in \mathbb{S}_{n,p,m}^0$. We use the notations of Theorem 3.1. Then S_+ solves (AP_2). More precisely, we have*

$$\inf_{\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+} \|\mathcal{G}(S) - \mathcal{G}(\widehat{S})\|_2 = \|\mathcal{G}(S) - \mathcal{G}(S_+)\|_2 = \|\mathcal{G}(S_-)\|_2\,.$$

The solution is unique in the following sense: If S_2 is another solution, then it is also a realization of $\mathcal{G}(S_+)$.

Proof. Let $\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+$ with $\mathcal{G}(S) - \mathcal{G}(\widehat{S}) \in RL_2$. Since $\mathcal{G}(S_-) \in RL_2$ (Theorem 2.19) we have $\mathcal{G}(S_+) - \mathcal{G}(\widehat{S}) \in RH_2$. By Theorem 2.18 we obtain

$$\|\mathcal{G}(S) - \mathcal{G}(\widehat{S})\|_2^2 = \|\mathcal{G}(S_-) + \mathcal{G}(S_+) - \mathcal{G}(\widehat{S})\|_2^2 = \|\mathcal{G}(S_-)\|_2^2 + \|\mathcal{G}(S_+) - \mathcal{G}(\widehat{S})\|_2^2 \geq \|\mathcal{G}(S_-)\|_2^2\,.$$

The lower bound is attained, if and only if $\widehat{S} \in \mathbb{S}(\mathcal{G}(S_+))$, e.g. if $\widehat{S} = S_+$. $\qquad\square$

Remark 3.3. In the proof of Theorem 3.2 it can be seen that the function $\mathcal{G}(S_+)$ is even the best approximation in the Hardy space H_2. The reason is that $\mathcal{G}(S_-)$ is also orthogonal to H_2 (see the more general statement of Theorem 2.18 in [Fra87, Section 2.3]).

3.3 Optimal RH_∞-approximations with balanced realization

For a minimal antistable standard system S the problem (AP_∞) was already solved in [Glo84]. Combining [Glo84, Theorem 6.1] and its proof based on [Glo84, Theorem 6.3], we obtain the following result. We reformulate the statements in terms of realizations instead of transfer functions. Note also the interchange of the stable and antistable system which is easy to show.

Theorem 3.4 ([Glo84]). *Let $S \in \mathbb{S}_{n,p,m}^-$ be a minimal realization such that*

$$S = \left(\begin{pmatrix} I & 0 \\ 0 & I_r \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D \right),$$

$$\mathfrak{C}_S = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & -\sigma_1 I_r \end{pmatrix}, \quad \mathfrak{D}_S = \begin{pmatrix} \Sigma_2 & 0 \\ 0 & -\sigma_1 I_r \end{pmatrix}, \quad \sigma_1^2 \notin \sigma(\Sigma_1 \Sigma_2),$$

where $\sigma_1 := \sqrt{\max \sigma(\mathfrak{C}_S \mathfrak{D}_S)}$ (compare Theorem 2.14). Then there exists $U \in \mathbb{R}^{p \times m}$ with $B_2 = -C_2^\top U$. Set

$$
\begin{aligned}
\Gamma &:= \Sigma_1 \Sigma_2 - \sigma_1^2 I, \\
\widetilde{A} &:= \Gamma^{-1}(\sigma_1^2 A_{11}^\top + \Sigma_2 A_{11} \Sigma_1 + \sigma_1 C_1^\top U B_1^\top), \quad \widetilde{C} := C_1 \Sigma_1 - \sigma_1 U B_1^\top, \\
\widetilde{B} &:= \Gamma^{-1}(\Sigma_2 B_1 - \sigma_1 C_1^\top U), \qquad\qquad\qquad\quad \widetilde{D} := D + \sigma_1 U.
\end{aligned}
$$

Then $\widetilde{S} := (I, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) \in \mathbb{S}_{n-r,p,m}^+$ holds and

$$\inf_{\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+} \| \mathcal{G}(S) - \mathcal{G}(\widehat{S}) \|_\infty = \| \mathcal{G}(S) - \mathcal{G}(\widetilde{S}) \|_\infty = \sigma_1.$$

Remark 3.5. The value σ_1 is the greatest *hankel singular value* of $\mathcal{G}(S)$ (see [Fra87, Section 5.1] for further details). As stated in [Glo84, Theorem 6.1], the function $\mathcal{G}(\widetilde{S})$ is even the best approximation in the more general Hardy space H_∞.

Because of Theorem 3.1 we are now able to solve (AP_∞), if S is a descriptor system. The disadvantage of algorithms based on Theorem 3.4 is that we need a balanced minimal realization of $\mathcal{G}(S_-)$ to compute an optimal solution.

3.4 Optimal RH_∞-approximations without balanced realization

Some results of [Glo84], concerning optimal hankel norm approximation, were generalized in [SCL90] to nonminimal stable standard systems. We use [SCL90, Theorem 1] and its proof to solve (AP_∞) for general $S \in \mathbb{S}_{n,p,m}^-$. Among some modifications, one of our main tasks is to prove that the optimal solution lies indeed in $\mathbb{S}_{n,p,m}^+$, since [SCL90] only considers the poles of the resulting transfer function.

First, we need to compute the greatest hankel singular value σ_1 of $\mathcal{G}(S)$ with the formula given in Theorem 3.4 which is restricted to minimal standard systems. However, we

avoid the determination of a minimal realization. To the author's knowledge, it was never shown that the nonzero hankel singular values of $\mathcal{G}(S)$ equal the roots of the nonzero eigenvalues of $\mathfrak{C}_S \mathfrak{D}_S$ independently of the realization S of $\mathcal{G}(S)$. This equality was only proved for minimal realizations (cf. e.g. [Fra87, Section 5.1, Theorem 3]).

Lemma 3.6. *Let* $S = (I, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$ *be a minimal realization of* $G \in \mathbb{T}_{p,m}$ *and let* $\widetilde{S} = (\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}) \in \mathbb{S}_{\widetilde{n},p,m}^-$ *be a realization of* G. *Then*

$$\sigma\big(\mathfrak{C}_S \mathfrak{D}_S\big) \setminus \{0\} = \sigma\big(\mathfrak{C}_{\widetilde{S}} \widetilde{E}^\top \mathfrak{D}_{\widetilde{S}} \widetilde{E}\big) \setminus \{0\} \,.$$

In particular, the equation $\sigma_1^2 = \max \sigma\big(\mathfrak{C}_{\widetilde{S}} \widetilde{E}^\top \mathfrak{D}_{\widetilde{S}} \widetilde{E}\big)$ *holds for all* $\widetilde{S} \in \mathbb{S}_{\widetilde{n},p,m}^- \cap \mathbb{S}(G)$.

Proof. We set $S_I := \widetilde{E}^{-1} \cdot \widetilde{S}$. By Theorem 2.11 there exists a regular $T \in \mathbb{R}^{\widetilde{n} \times \widetilde{n}}$ such that

$$S_T := T \cdot S_I \cdot T^{-1} = \left(I, \begin{pmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} C_1 & 0 & C_3 & 0 \end{pmatrix}, \widetilde{D} \right),$$

where $S_1 := (I, A_{11}, B_1, C_1, \widetilde{D})$ is a minimal realization of G. The condition $\sigma(\widetilde{E}, \widetilde{A}) = \sigma(\widetilde{E}^{-1}\widetilde{A}) \subseteq \mathbb{C}_{>0}$ implies

$$\sigma\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \subseteq \mathbb{C}_{>0}, \qquad \sigma\begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix} \subseteq \mathbb{C}_{>0}, \qquad \sigma(A_{11}) \subseteq \mathbb{C}_{>0}.$$

By Theorem 2.12 there exist unique solutions $\begin{pmatrix} X_1 & X_2 \\ X_2^\top & X_4 \end{pmatrix}$ and $\begin{pmatrix} Y_1 & Y_2 \\ Y_2^\top & Y_4 \end{pmatrix}$ of

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} X_1 & X_2 \\ X_2^\top & X_4 \end{pmatrix} + \begin{pmatrix} X_1 & X_2 \\ X_2^\top & X_4 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}^\top + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{pmatrix} B_1^\top & B_2^\top \end{pmatrix} = 0,$$

$$\begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix}^\top \begin{pmatrix} Y_1 & Y_2 \\ Y_2^\top & Y_4 \end{pmatrix} + \begin{pmatrix} Y_1 & Y_2 \\ Y_2^\top & Y_4 \end{pmatrix} \begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix} + \begin{pmatrix} C_1^\top \\ C_3^\top \end{pmatrix} \begin{pmatrix} C_1 & C_3 \end{pmatrix} = 0.$$

Thus $X_1 = \mathfrak{C}_{S_1}$, $Y_1 = \mathfrak{D}_{S_1}$,

$$\mathfrak{C}_{S_T} = \begin{pmatrix} X_1 & X_2 & 0 & 0 \\ X_2^\top & X_4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \mathfrak{D}_{S_T} = \begin{pmatrix} Y_1 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 0 \\ Y_2^\top & 0 & Y_4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 2.11 and Lemma 2.13 there exists a regular $P \in \mathbb{R}^{n \times n}$ with $\mathfrak{C}_{S_1} = P^{-1} \mathfrak{C}_S P^{-\top}$,

$\mathfrak{D}_{S_1} = P^\top \mathfrak{D}_S P$. We conclude

$$\sigma\big(\mathfrak{C}_{\widetilde{S}}\widetilde{E}^\top \mathfrak{D}_{\widetilde{S}}\widetilde{E}\big)\setminus\{0\} = \sigma\big(\mathfrak{C}_{S_I}\mathfrak{D}_{S_I}\big)\setminus\{0\} = \sigma\big(T\mathfrak{C}_{S_T}T^\top T^{-\top}\mathfrak{D}_{S_T}T^{-1}\big)\setminus\{0\}$$

$$= \sigma\big(\mathfrak{C}_{S_T}\mathfrak{D}_{S_T}\big)\setminus\{0\} = \sigma\begin{pmatrix} X_1Y_1 & 0 & X_1Y_2 & 0 \\ X_2^\top Y_1 & 0 & X_2^\top Y_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\setminus\{0\}$$

$$= \sigma\big(X_1Y_1\big)\setminus\{0\} = \sigma\big(\mathfrak{C}_{S_1}\mathfrak{D}_{S_1}\big)\setminus\{0\} = \sigma\big(\mathfrak{C}_S\mathfrak{D}_S\big)\setminus\{0\}\,. \qquad \square$$

The next theorem is one of our main results. Together with Theorem 3.1 it gives suboptimal solutions of (AP_∞) and, with an additional regularity-condition, an optimal solution. Theorem 3.14 solves (AP_∞), if the regularity-condition is not fulfilled.

Theorem 3.7. *Let $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$. Choose $\gamma \geq \sigma_1$ and set*

$$\mathcal{R}_{S,\gamma} := \mathfrak{D}_S E \mathfrak{C}_S E^\top - \gamma^2 I\,, \quad \mathcal{E}_{S,\gamma} := E^\top \mathcal{R}_{S,\gamma}\,, \qquad\qquad \mathcal{B}_{S,\gamma} := E^\top \mathfrak{D}_S B\,,$$

$$\mathcal{A}_{S,\gamma} := -A^\top \mathcal{R}_{S,\gamma} - C^\top \mathcal{C}_{S,\gamma}\,, \qquad \mathcal{C}_{S,\gamma} := C\mathfrak{C}_S E^\top\,.$$

If $(\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma})$ is regular, e.g. if $\gamma > \sigma_1$, then $\mathcal{S}_{S,\gamma} := (\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma}, \mathcal{B}_{S,\gamma}, \mathcal{C}_{S,\gamma}, D) \in \mathbb{S}_{n,p,m}^+$ and $\mathcal{S}_{S,\gamma}$ fulfils

$$\sigma_1 \leq \|\mathcal{G}(S) - \mathcal{G}(\mathcal{S}_{S,\gamma})\|_\infty \leq \gamma\,.$$

Proof. First, let $E = I$. In this and the following proofs we use the equation

$$\mathcal{A}_{S,\gamma} = -A^\top \mathcal{R}_{S,\gamma} - C^\top \mathcal{C}_{S,\gamma} = -\mathcal{R}_{S,\gamma}A^\top - \mathcal{B}_{S,\gamma}B^\top = \gamma^2 A^\top + \mathfrak{D}_S A\mathfrak{C}_S\,, \qquad (3.1)$$

which holds because of

$$-A^\top \mathcal{R}_{S,\gamma} - C^\top \mathcal{C}_{S,\gamma} = \gamma^2 A^\top - A^\top \mathfrak{D}_S\mathfrak{C}_S - C^\top C\mathfrak{C}_S = \gamma^2 A^\top + \mathfrak{D}_S A\mathfrak{C}_S$$

$$= \gamma^2 A^\top - \mathfrak{D}_S\mathfrak{C}_S A^\top - \mathfrak{D}_S BB^\top = -\mathcal{R}_{S,\gamma}A^\top - \mathcal{B}_{S,\gamma}B^\top\,.$$

Thus $\mathcal{A}_{S,\gamma}$ is indeed the same as in [SCL90, Theorem 1]. To apply [SCL90, Theorem 1], we first construct an antistable system S_2 with a hankel singular value greater than γ. This method was proposed in [Glo84, Remark 8.4]. Let $\gamma_1 > \gamma$. We set $S_1 := (1, \frac{1}{2}\gamma_1, \gamma_1, \gamma_1, 0)$ and

$$S_2 := \left(I, \begin{pmatrix} A & 0 \\ 0 & \frac{1}{2}\gamma_1 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \gamma_1 \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & \gamma_1 \end{pmatrix}, \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\right) \in \mathbb{S}_{n+1,p+1,m+1}^-\,.$$

Then we have

$$\mathfrak{C}_{S_2} = \begin{pmatrix} \mathfrak{C}_S & 0 \\ 0 & -\gamma_1 \end{pmatrix}, \quad \mathfrak{D}_{S_2} = \begin{pmatrix} \mathfrak{D}_S & 0 \\ 0 & -\gamma_1 \end{pmatrix}\,.$$

Thus γ_1 is the greatest hankel singular value of $\mathcal{G}(S_2)$ and $\sigma_1 \leq \gamma < \gamma_1$. If we apply [SCL90, Theorem 1] to S_2 with $K = 0 \in RH_\infty$, then the transfer function of

$$\mathcal{S}_{S_2,\gamma} = \left(\begin{pmatrix} \mathcal{E}_{S,\gamma} & 0 \\ 0 & \gamma_1^2 - \gamma^2 \end{pmatrix}, \begin{pmatrix} \mathcal{A}_{S,\gamma} & 0 \\ 0 & \frac{1}{2}(\gamma^2\gamma_1 + \gamma_1^3) \end{pmatrix}, \begin{pmatrix} \mathcal{B}_{S,\gamma} & 0 \\ 0 & -\gamma_1^2 \end{pmatrix}, \begin{pmatrix} \mathcal{C}_{S,\gamma} & 0 \\ 0 & -\gamma_1^2 \end{pmatrix}, \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\right)$$

is a suboptimal solution, i.e. $\left\|\mathcal{G}(S_2) - \mathcal{G}(\mathcal{S}_{S_2,\gamma})\right\|_\infty \leq \gamma$. Note that [SCL90, Theorem 1] does not state, that for every $K \in RH_\infty$ with $\|K\|_\infty \leq 1$ the resulting transfer function is a suboptimal solution. However, this was shown in the proof (see also [Glo84, Remark 8.4]). With

$$\mathcal{G}(S_2) = \begin{pmatrix} \mathcal{G}(S) & 0 \\ 0 & \mathcal{G}(S_1) \end{pmatrix}, \quad \mathcal{G}(\mathcal{S}_{S_2,\gamma}) = \begin{pmatrix} \mathcal{G}(\mathcal{S}_{S,\gamma}) & 0 \\ 0 & \mathcal{G}(\mathcal{S}_{S_1,\gamma}) \end{pmatrix}$$

we conclude

$$\left\|\mathcal{G}(S) - \mathcal{G}(\mathcal{S}_{S,\gamma})\right\|_\infty \leq \left\|\mathcal{G}(S_2) - \mathcal{G}(\mathcal{S}_{S_2,\gamma})\right\|_\infty \leq \gamma\,.$$

Now we show that $\mathcal{S}_{S,\gamma} \in \mathbb{S}_{n,p,m}^+$. [SCL90, Theorem 1] states that $\mathcal{G}(\mathcal{S}_{S_2,\gamma})$ has at most one pole λ in $\mathbb{C}_{\geq 0}$ counting multiplicities. The pole of $\mathcal{G}(\mathcal{S}_{S_1,\gamma})$ is $\widetilde{\lambda} = \frac{2(\gamma_1^2 - \gamma^2)}{\gamma^2 \gamma_1 + \gamma_1^3} > 0$ with $\gamma_1^2 - \gamma^2 > 0$, $\gamma^2 \gamma_1 + \gamma_1^3 > 0$. Thus $\lambda = \widetilde{\lambda}$ and $\mathcal{G}(\mathcal{S}_{S,\gamma})$ has no poles in $\mathbb{C}_{\geq 0}$.

Let $\lambda \in \mathbb{C}_{\geq 0}$. We use Lemma 2.15 to show that $\lambda \notin \sigma(\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma})$. Let $v \in \mathbb{R}^n$ satisfy $(\lambda \mathcal{E}_{S,\gamma} - \mathcal{A}_{S,\gamma})v = 0$ and $\mathcal{C}_{S,\gamma} v = 0$. Hence with $\mathcal{A}_{S,\gamma} = -A^\top \mathcal{E}_{S,\gamma} - C^\top \mathcal{C}_{S,\gamma}$ we get $(\lambda I - (-A^\top))\mathcal{E}_{S,\gamma} v = 0$. Because of $\sigma(-A^\top) \subseteq \mathbb{C}_{<0}$ we conclude $\mathcal{E}_{S,\gamma} v = 0$. Thus

$$\forall\, s \in \mathbb{C}\colon (s\mathcal{E}_{S,\gamma} - \mathcal{A}_{S,\gamma})v = (s\mathcal{E}_{S,\gamma} + A^\top \mathcal{E}_{S,\gamma} + C^\top \mathcal{C}_{S,\gamma})v = 0\,.$$

Since the matrix pair $(\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma})$ is regular we have $v = 0$.

Now let $v \in \mathbb{R}^n$ satisfy $(\lambda \mathcal{E}_{S,\gamma}^\top - \mathcal{A}_{S,\gamma}^\top)v = 0$ and $\mathcal{B}_{S,\gamma}^\top v = 0$. Hence $(\lambda I + A)\mathcal{E}_{S,\gamma}^\top v = 0$ follows with $\mathcal{A}_{S,\gamma}^\top = -A\mathcal{E}_{S,\gamma}^\top - B\mathcal{B}_{S,\gamma}^\top$. Again we deduce $v = 0$. Therefore $\lambda \notin \sigma(\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma})$ and thus $\mathcal{S}_{S,\gamma} \in \mathbb{S}_{n,p,m}^+$ follows.

Finally we consider general regular $E \in \mathbb{R}^{n \times n}$. We already proved that the assertion holds for $\mathcal{S}_{E^{-1}\cdot S,\gamma}$. With $\mathfrak{C}_{E^{-1}\cdot S} = \mathfrak{C}_S$ and $\mathfrak{D}_{E^{-1}\cdot S} = E^\top \mathfrak{D}_S E$ (Lemma 2.13) we conclude

$$\mathcal{E}_{E^{-1}\cdot S,\gamma} = E^\top \mathfrak{D}_S E \mathfrak{C}_S - \gamma^2 I\,, \qquad\qquad \mathcal{B}_{E^{-1}\cdot S,\gamma} = E^\top \mathfrak{D}_S E E^{-1} B\,,$$

$$\mathcal{A}_{E^{-1}\cdot S,\gamma} = -A^\top E^{-\top} \mathcal{E}_{E^{-1}\cdot S,\gamma} - C^\top \mathcal{C}_{E^{-1}\cdot S,\gamma}\,, \quad \mathcal{C}_{E^{-1}\cdot S,\gamma} = C\mathfrak{C}_S\,.$$

Now we see that $\mathcal{S}_{S,\gamma} = \mathcal{S}_{E^{-1}\cdot S,\gamma} \cdot E^\top$. Thus the assertion also holds for $\mathcal{S}_{S,\gamma} \sim \mathcal{S}_{E^{-1}\cdot S,\gamma}$. If $\gamma > \sigma_1$, the matrix pair $(\mathcal{E}_{S,\gamma}, \mathcal{A}_{S,\gamma})$ is regular, since $\mathcal{R}_{S,\gamma}$, E and thus $\mathcal{E}_{S,\gamma}$ are regular. $\qquad\square$

We formulate some equivalent conditions for the regularity-condition of $(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$ in Theorem 3.7.

Corollary 3.8. *The following three statements are equivalent.*

(i) $(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$ *is regular,* *(ii)* $\sigma(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}) \subseteq \mathbb{C}_{<0} \cup \{\infty\}$, *(iii)* $\mathcal{A}_{S,\sigma_1}$ *is regular*.

Proof. The implication "(i) $\Rightarrow$ (ii)" follows from Theorem 3.7. The statement (ii) implies $0 \notin \sigma(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$ and thus $(0 \cdot \mathcal{E}_{S,\sigma_1} + \mathcal{A}_{S,\sigma_1})$ is regular. Finally, (iii) implies $0 \notin \sigma(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$, therefore $\rho(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}) \neq \emptyset$ and thus (i) holds. $\qquad\square$

To get optimal solutions if $\left(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}\right)$ is singular, we analyze $\mathcal{S}_{S,\gamma}$ in Theorem 3.7 for $\gamma \to \sigma_1+$. For the proofs we use the realization $S_b \sim S$ of $\mathcal{G}(S)$ as in Theorem 2.14 with $h = \sigma_1$ because we will take advantage of the special structure of $\mathcal{S}_{S_b,\sigma_1}$. First we investigate the relation between $\mathcal{S}_{S,\gamma}$ and $\mathcal{S}_{S_b,\gamma}$.

Lemma 3.9. *Let* $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$. *Let* $P, Q \in \mathbb{R}^{n \times n}$ *be regular. Then* $\mathcal{S}_{P \cdot S \cdot Q, \gamma} = Q^T \cdot \mathcal{S}_{S,\gamma} \cdot P^T$ *holds for all* $\gamma \in \mathbb{R}$.

Proof. We set $\widetilde{S} := P \cdot S \cdot Q$. By Lemma 2.13 the equations $\mathfrak{C}_S = Q\mathfrak{C}_{\widetilde{S}}Q^\top$, $\mathfrak{D}_S = P^\top\mathfrak{D}_{\widetilde{S}}P$ hold. This implies

$$
\begin{aligned}
\mathcal{B}_{S,\gamma} &= E^\top\mathfrak{D}_S B = E^\top P^\top\mathfrak{D}_{\widetilde{S}}PB = Q^{-\top}\mathcal{B}_{\widetilde{S},\gamma}\,, \\
\mathcal{C}_{S,\gamma} &= C\mathfrak{C}_S E^\top = CQ\mathfrak{C}_{\widetilde{S}}Q^\top E^\top = \mathcal{C}_{\widetilde{S},\gamma}P^{-\top}, \\
\mathcal{E}_{S,\gamma} &= E^\top\mathfrak{D}_S E\mathfrak{C}_S E^\top - \gamma^2 E^\top = E^\top P^\top\mathfrak{D}_{\widetilde{S}}PEQ\mathfrak{C}_{\widetilde{S}}Q^\top E^\top - \gamma^2 E^\top = Q^{-\top}\mathcal{E}_{\widetilde{S},\gamma}P^{-\top}, \\
\mathcal{A}_{S,\gamma} &= -A^\top E^{-\top}\mathcal{E}_{S,\gamma} + C^\top\mathcal{C}_{S,\gamma} = -A^\top E^{-\top}Q^{-\top}\mathcal{E}_{\widetilde{S},\gamma}P^{-\top} + C^\top\mathcal{C}_{\widetilde{S},\gamma}P^{-\top} \\
&= Q^{-\top}\mathcal{A}_{\widetilde{S},\gamma}P^{-\top}. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Lemma 3.10. *Let* $S_b = (I, A_b, B_b, C_b, D) \sim S$ *be as in Theorem 2.14 with* $h = \sigma_1$. *Let* $T_1 \in \mathbb{R}^{n \times n}$ *be regular such that*

$$
T_1 \cdot S \cdot E^{-1}T_1^{-1} = S_b = \left(\begin{pmatrix} I & 0 \\ 0 & I_r \end{pmatrix}, \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \begin{pmatrix} C_1 & C_2 \end{pmatrix}, D\right).
$$

Then the following three statements hold:

(a) $B_2 B_2^\top = C_2^\top C_2$,

(b) $\ker(B_2^\top) = \ker(C_2)$,

(c) $\left(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}\right)$ *singular* $\Leftrightarrow \ker(C_2) \neq \{0\} \Leftrightarrow \ker(B_2^\top) \neq \{0\}$.

Proof. We get $B_2 B_2^\top = C_2^\top C_2$ by adding the equations $-\sigma_1 A_{22} - \sigma_1 A_{22}^\top + B_2 B_2^\top = 0$ and $\sigma_1 A_{22}^\top + \sigma_1 A_{22} - C_2^\top C_2 = 0$ resulting from (2.4). Therefore

$$
v \in \ker(B_2^\top) \Leftrightarrow B_2^\top v = 0 \Leftrightarrow \left\|B_2^\top v\right\|_2 = 0 \Leftrightarrow 0 = v^\top B_2 B_2^\top v = v^\top C_2^\top C_2 v
$$

$$
\Leftrightarrow \|C_2 v\|_2 = 0 \Leftrightarrow v \in \ker(C_2).
$$

By Lemma 3.9 and with $\Gamma := \Sigma_2\Sigma_1 - \sigma_1^2 I$ we have

$$
\begin{aligned}
\mathcal{S}_{S_b,\sigma_1} &= T_1^{-\top}E^{-\top} \cdot \mathcal{S}_{S,\sigma_1} \cdot T_1^\top \\
&= \left(\begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -A_{11}^\top\Gamma - C_1^\top C_1\Sigma_1 & \sigma_1 C_1^\top C_2 \\ -A_{12}^\top\Gamma - C_2^\top C_1\Sigma_1 & \sigma_1 C_2^\top C_2 \end{pmatrix}, \begin{pmatrix} \Sigma_2 B_1 \\ -\sigma_1 B_2 \end{pmatrix}, \begin{pmatrix} C_1\Sigma_1 & -\sigma_1 C_2 \end{pmatrix}, 0\right).
\end{aligned}
$$

Therefore $\left(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}\right)$ is singular, if and only if $\left(\mathcal{E}_{S_b,\sigma_1}, \mathcal{A}_{S_b,\sigma_1}\right)$ is singular.

Let $\ker(C_2) \neq \{0\}$. Then there exists $v \neq 0$ with $C_2 v = 0$, i.e. for all $s \in \mathbb{C}$

$$(s\mathcal{E}_{S_b,\sigma_1} - \mathcal{A}_{S_b,\sigma_1})\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} s\Gamma + A_{11}^\top\Gamma + C_1^\top C_1 \Sigma_1 & -\sigma_1 C_1^\top C_2 \\ A_{12}^\top\Gamma + C_2^\top C_1 \Sigma_1 & -\sigma_1 C_2^\top C_2 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

That means $(\mathcal{E}_{S_b,\sigma_1}, \mathcal{A}_{S_b,\sigma_1})$ is singular.

Let $\ker(C_2) = \{0\}$. Then $C_2^\top C_2$ is regular. Let $s \in \mathbb{C}$. The matrix $s\mathcal{E}_{S_b,\sigma_1} - \mathcal{A}_{S_b,\sigma_1}$ is singular, if and only if the Schur complement

$$\left(sI + \left(A_{11}^\top + C_1^\top C_1 \Sigma_1 \Gamma^{-1} - C_1^\top C_2 (C_2^\top C_2)^{-1}(A_{12}^\top\Gamma + C_2^\top C_1 \Sigma_1)\Gamma^{-1}\right)\right)\Gamma =: (sI - M)\Gamma$$

of $s\mathcal{E}_{S_b,\sigma_1} - \mathcal{A}_{S_b,\sigma_1}$ in $-\sigma_1 C_2^\top C_2$ is singular (Theorem 2.5). This is equivalent to $s \in \sigma(M)$. Hence $(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$ has a finite number of eigenvalues and consequently is regular. $\quad\square$

Similar to the proof of [SCL90, Theorem 1] we want to eliminate the singular part of the system $\mathcal{S}_{S,\sigma_1}$ and show that the resulting system is an optimal solution.

Lemma 3.11. *Let $(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1})$ be singular. Then there exists a regular $T \in \mathbb{R}^{n \times n}$ such that*

$$TE^{-\top} \cdot \mathcal{S}_{S,\sigma_1} \cdot T^{-1} = \left(\begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widetilde{B}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix}, D\right),$$

where $(\widetilde{E}_{11}, \widetilde{A}_{11})$ is regular.

Proof. Our proof is based on the proof of [SCL90, Theorem 1]. Let S_b, T_1 and Γ be as in Lemma 3.10. By Lemma 3.10 we have $d := \dim(\ker(C_2)) > 0$. Therefore there exists an orthogonal $W \in \mathbb{R}^{r \times r}$ such that $(\widehat{C}_2 \ 0_{p \times d}) = C_2 W^\top$ and $\ker(\widehat{C}_2) = \{0\}$ (e.g. singular value decomposition). We define $(\widehat{B}_2^\top \ \widehat{B}_3^\top) := B_2^\top W^\top$ with appropriate dimensions. By Lemma 3.10 we have $\widehat{B}_3^\top = 0$. Now we set $T_2 := \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix}$, conclude

$-A_{12}^\top \Gamma - C_2^\top C_1 \Sigma_1 = \sigma_1 B_2 B_1^\top$ from (3.1) and compute

$$T_2 \mathcal{B}_{S_b,\sigma_1} = T_2 \mathfrak{C}_{S_b} B_b = \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Sigma_2 B_1 \\ -\sigma_1 B_2 \end{pmatrix} = \begin{pmatrix} \Sigma_2 B_1 \\ -\sigma_1 \widehat{B}_2 \\ 0_{d \times m} \end{pmatrix} =: \begin{pmatrix} \widetilde{B}_1 \\ 0_{d \times m} \end{pmatrix}$$

$$\mathcal{C}_{S_b,\sigma_1} T_2^{-1} = C_b \mathfrak{D}_{S_b} T_2^{-1} = \begin{pmatrix} C_1 \Sigma_1 & -\sigma_1 C_2 W^\top \end{pmatrix}$$

$$= \begin{pmatrix} C_1 \Sigma_1 & -\sigma_1 \widehat{C}_2 & 0_{p \times d} \end{pmatrix} =: \begin{pmatrix} \widetilde{C}_1 & 0_{p \times d} \end{pmatrix}$$

$$T_2 \mathcal{E}_{S_b,\sigma_1} T_2^{-1} = \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & W^\top \end{pmatrix} = \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{d \times d} \end{pmatrix} =: \begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0_{d \times d} \end{pmatrix}$$

$$T_2 \mathcal{A}_{S_b,\sigma_1} T_2^{-1} = T_2 \begin{pmatrix} -A_{11}^\top \Gamma - C_1^\top C_1 \Sigma_1 & \sigma_1 C_1^\top C_2 \\ \sigma_1 B_2 B_1^\top & \sigma_1 C_2^\top C_2 \end{pmatrix} T_2^{-1}$$

$$= \begin{pmatrix} -A_{11}^\top \Gamma - C_1^\top C_1 \Sigma_1 & \sigma_1 C_1^\top \widehat{C}_2 & 0 \\ \sigma_1 \widehat{B}_2 B_1^\top & \sigma_1 \widehat{C}_2^\top \widehat{C}_2 & 0 \\ 0 & 0 & 0_{d \times d} \end{pmatrix} =: \begin{pmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0_{d \times d} \end{pmatrix}.$$

Since $\widehat{C}_2^\top \widehat{C}_2$ is regular, there exists $s \in \mathbb{C}$ such that the Schur complement

$$\left(sI + \left(A_{11}^\top + C_1^\top C_1 \Sigma_1 \Gamma^{-1} - \sigma_1 C_1^\top \widehat{C}_2 (\widehat{C}_2^\top \widehat{C}_2)^{-1} \widehat{B}_2 B_1^\top \Gamma^{-1} \right) \right) \Gamma$$

of $s\widetilde{E}_{11} - \widetilde{A}_{11}$ in $-\sigma_1 \widehat{C}_2^\top \widehat{C}_2$ is regular. Theorem 2.5 states that $s\widetilde{E}_{11} - \widetilde{A}_{11}$ is regular. Thus $(\widetilde{E}_{11}, \widetilde{A}_{11})$ is regular. With $T := T_2 T_1^{-\top}$ the proof is finished. $\qquad \square$

Lemma 3.12. *We set* $S_1 := (\widetilde{E}_{11}, \widetilde{A}_{11}, \widetilde{B}_1, \widetilde{C}_1, D)$. *For every* $\omega \in \mathbb{R}$ *with* $\mathrm{i}\omega \in \rho(\widetilde{E}_{11}, \widetilde{A}_{11})$ *the unequality* $\|\mathcal{G}(S)(\mathrm{i}\omega) - \mathcal{G}(S_1)(\mathrm{i}\omega)\|_2 \leq \sigma_1$ *holds. Moreover,* $\mathcal{G}(S_1)$ *has no poles in* $\mathbb{C}_{\geq 0}$.

Proof. First we show that $\mathcal{G}(\mathcal{S}_{S,\gamma})(s) \xrightarrow{\gamma \to \sigma_1 +} \mathcal{G}(S_1)(s)$ for all $s \in \mathbb{C}_{\geq 0}$ except for a finite number. Let T be as in Lemma 3.11. We rewrite $\mathcal{G}(\mathcal{S}_{S,\gamma})(s)$ for $\gamma > \sigma_1$ and $s \in \mathbb{C}_{\geq 0}$ as

$$\mathcal{G}(\mathcal{S}_{S,\gamma})(s) = \mathcal{G}(TE^{-\top} \cdot \mathcal{S}_{S,\gamma} \cdot T^{-1})(s) = \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix} \left(TE^{-\top}(s\mathcal{E}_{S,\gamma} - \mathcal{A}_{S,\gamma})T^{-1} \right)^{-1} \begin{pmatrix} \widetilde{B}_1 \\ 0 \end{pmatrix} + D$$

$$\overset{(*)}{=} \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix} \left(\begin{pmatrix} s\widetilde{E}_{11} - \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} + (\sigma_1^2 - \gamma^2)(sI + TE^{-\top}A^\top T^{-1}) \right)^{-1} \begin{pmatrix} \widetilde{B}_1 \\ 0 \end{pmatrix} + D,$$

where $(*)$ holds with

$$\mathcal{E}_{S,\gamma} = \mathcal{E}_{S,\sigma_1} + \mathcal{E}_{S,\gamma} - \mathcal{E}_{S,\sigma_1} = \mathcal{E}_{S,\sigma_1} + (\sigma_1^2 - \gamma^2)E^\top,$$

$$\mathcal{A}_{S,\gamma} = \mathcal{A}_{S,\sigma_1} - (\sigma_1^2 - \gamma^2)A^\top.$$

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We set $\left(\begin{smallmatrix} M_1 & M_2 \\ M_3 & M_4 \end{smallmatrix} \right) := TE^{-\top}A^\top T^{-1}$ with appropriate dimensioned M_i. In addition, let $s \notin \sigma(M_4) \cup \sigma(\widetilde{E}_{11}, \widetilde{A}_{11})$. Then

$$L_\gamma := s\widetilde{E}_{11} - \widetilde{A}_{11} + (\sigma_1^2 - \gamma^2)(sI + M_1) - (\sigma_1^2 - \gamma^2)M_2(sI - M_4)^{-1}M_3$$

is the Schur complement of

$$TE^{-\top}(s\mathcal{E}_{s,\gamma} - \mathcal{A}_{s,\gamma})T^{-1} = \begin{pmatrix} s\widetilde{E}_{11} - \widetilde{A}_{11} + (\sigma_1^2 - \gamma^2)(sI + M_1) & (\sigma_1^2 - \gamma^2)M_2 \\ (\sigma_1^2 - \gamma^2)M_3 & (\sigma_1^2 - \gamma^2)(sI + M_4) \end{pmatrix}$$

in $(\sigma_1^2 - \gamma^2)(sI + M_4)$. By [Theorem 2.5](#) the matrix L_γ is regular because $(s\mathcal{E}_{s,\gamma} - \mathcal{A}_{s,\gamma})$ is regular. The functions $\gamma \mapsto L_\gamma$ and thus $\gamma \mapsto \mathcal{G}(\mathcal{S}_{s,\gamma})(s) = \widetilde{C}_1 L_\gamma^{-1}\widetilde{B}_1 + D$ are continuous in (σ_1, ∞). Note that $L_{\sigma_1} = s\widetilde{E}_{11} - \widetilde{A}_{11}$ is regular. We obtain

$$\mathcal{G}(\mathcal{S}_{s,\gamma})(s) = \widetilde{C}_1 L_\gamma^{-1}\widetilde{B}_1 + D \xrightarrow{\gamma \to \sigma_1+} \widetilde{C}_1(s\widetilde{E}_{11} - \widetilde{A}_{11})^{-1}\widetilde{B}_1 + D = \mathcal{G}(S_1)(s).$$

Therefore for all $i\omega \notin \sigma(M_4) \cup \sigma(\widetilde{E}_{11}, \widetilde{A}_{11})$

$$\|\mathcal{G}(S)(i\omega) - \mathcal{G}(S_1)(i\omega)\|_2 = \lim_{\gamma \to \sigma_1+} \|\mathcal{G}(S)(i\omega) - \mathcal{G}(S_{S,\gamma})(i\omega)\|_2 \le \lim_{\gamma \to \sigma_1+} \gamma = \sigma_1.$$

Now we show that $\mathcal{G}(S_1)$ has no poles in $\mathbb{C}_{\ge 0}$. By [Fra87, Section 2.3] we have $\|G(s)\|_2 \le \|G\|_\infty$ for all $G \in RH_\infty$ and $s \in \mathbb{C}_{\ge 0}$. Thus we get the upper estimate

$$\|\mathcal{G}(\mathcal{S}_{s,\gamma})(s)\|_2 \le \|\mathcal{G}(\mathcal{S}_{s,\gamma})(s) - \mathcal{G}(S)(s)\|_2 + \|\mathcal{G}(S)(s)\|_2 \le \gamma + \|\mathcal{G}(S)\|_\infty.$$

for all $s \in \mathbb{C}_{\ge 0}$. Hence for all $s \in \mathbb{C}_{\ge 0}\backslash(\sigma(\widetilde{E}_{11}, \widetilde{A}_{11}) \cup \sigma(M_4))$ we have

$$\|\mathcal{G}(S_1)(s)\|_2 = \lim_{\gamma \to \sigma_1+} \|\mathcal{G}(\mathcal{S}_{s,\gamma})(s)\|_2 \le \lim_{\gamma \to \sigma_1+}(\gamma + \|\mathcal{G}(S)\|_\infty) = \sigma_1 + \|\mathcal{G}(S)\|_\infty.$$

Thus $\mathcal{G}(S_1)$ has no poles in $\mathbb{C}_{\ge 0}$. $\qquad\square$

Theorem 3.13. *The system S_1 given in [Lemma 3.12](#) solves [(AP$_\infty$)](#).*

Proof. We use the notations of the proof of [Lemma 3.11](#). By [Theorem 3.7](#) and [Lemma 3.12](#) it remains to prove that $\sigma(\widetilde{E}_{11}, \widetilde{A}_{11}) \subseteq \mathbb{C}_{<0} \cup \{\infty\}$. Let $s_0 \in \mathbb{C}_{\ge 0}$. We apply [Lemma 2.15](#) because s_0 is not a pole of $\mathcal{G}(S_1)$. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfy $0 = \widetilde{C}_1 x$ and

$$0 = (s_0\widetilde{E}_{11} - \widetilde{A}_{11})x = \begin{pmatrix} s_0\Gamma + A_{11}^\top\Gamma + C_1^\top C_1\Sigma_1 & -\sigma_1 C_1^\top\widehat{C}_2 \\ -\sigma_1\widehat{B}_2 B_1^\top & -\sigma_1\widehat{C}_2^\top\widehat{C}_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad (3.2)$$

Hence

$$0 = \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \mathcal{C}_{S_b,\sigma_1} T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix},$$

(3.3)

$$0 = \begin{pmatrix} s_0 \widetilde{E}_{11} - \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = T_2(s_0 \mathcal{E}_{S_b,\sigma_1} - \mathcal{A}_{S_b,\sigma_1}) T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$= T_2(s_0 \mathcal{E}_{S_b,\sigma_1} + A_b^\top \mathcal{E}_{S_b,\sigma_1} + C_b^\top \mathcal{C}_{S_b,\sigma_1}) T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix} \overset{(3.3)}{=} T_2(s_0 I + A_b^\top) \mathcal{E}_{S_b,\sigma_1} T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Due to $S_b \sim S \in \mathbb{S}_{n,p,m}^-$ we have $s_0 \notin \sigma(-A_b^\top)$ and the equation

$$0 = \mathcal{E}_{S_b,\sigma_1} T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix} = T_2 \mathcal{E}_{S_b,\sigma_1} T_2^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} \Gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

follows. Thus we get $x_1 = 0$ by the regularity of Γ. Now we deduce

$$(3.2) \Rightarrow \sigma_1 \widehat{C}_2^\top \widehat{C}_2 x_2 = 0 \Rightarrow x_2 = 0$$

from the regularity of $\widehat{C}_2^\top \widehat{C}_2$. Thus $x = 0$. Analogously every $x \in \mathbb{R}^{n-d}$ which fulfils $0 = (s_0 \widetilde{E}_{11} - \widetilde{A}_{11})^\top x$ and $0 = \widetilde{B}_1^\top x$ equals zero. Thus $s_0 \notin \sigma(\widetilde{E}_{11}, \widetilde{A}_{11})$ and $\sigma(\widetilde{E}_{11}, \widetilde{A}_{11}) \subseteq \mathbb{C}_{<0} \cup \{\infty\}$. $\qquad\square$

The optimal solution given in the theorem above is still based on the balanced realization. Our second main contribution is the following result, which only needs one singular value decomposition.

Theorem 3.14. *Let $S = (E, A, B, C, D) \in \mathbb{S}_{n,p,m}^-$. Suppose $\left(\mathcal{E}_{S,\sigma_1}, \mathcal{A}_{S,\sigma_1}\right)$ given in Theorem 3.7 is singular. Then there exist orthogonal $U, V \in \mathbb{R}^{n \times n}$ with*

$$U \mathcal{A}_{S,\sigma_1} V = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & 0 \end{pmatrix} \text{ and regular } \widehat{A}_{11}.$$

(3.4)

Every regular $U, V \in \mathbb{R}^{n \times n}$ satisfying (3.4), fulfil

$$U \mathcal{E}_{S,\sigma_1} V = \begin{pmatrix} \widehat{E}_{11} & \widehat{E}_{12} \\ 0 & 0 \end{pmatrix}, \qquad U \mathcal{B}_{S,\sigma_1} = \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix}, \qquad \mathcal{C}_{S,\sigma_1} V = \begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix}.$$

The system $S_2 := \left(\widehat{E}_{11}, \widehat{A}_{11}, \widehat{B}_1, \widehat{C}_1, D\right)$ solves (AP$_\infty$).

Proof. Let $T \in \mathbb{R}^{n \times n}$ be as in Lemma 3.11. Then we obtain

$$\left(T E^{-\top} \mathcal{E}_{S,\sigma_1} T^{-1}, T E^{-\top} \mathcal{A}_{S,\sigma_1} T^{-1}\right) = \left(\begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix}\right)$$

with $\widetilde{A}_{11} \in \mathbb{R}^{(n-d)\times(n-d)}$ and $\sigma(\widetilde{E}_{11}, \widetilde{A}_{11}) \subseteq \mathbb{C}_{<0} \cup \{\infty\}$. In particular, $\widetilde{A}_{11}$ is regular. The matrices $\mathcal{A}_{s,\sigma_1}$ and $TE^{-\top}\mathcal{A}_{s,\sigma_1}T^{-1}$ have the same rank. By singular value decomposition there exist regular orthogonal $U, V \in \mathbb{R}^{n\times n}$ such that

$$U\mathcal{A}_{s,\sigma_1}V = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & 0 \end{pmatrix} \text{ with regular } \widehat{A}_{11} \in \mathbb{R}^{(n-d)\times(n-d)}.$$

Let $U, V \in \mathbb{R}^{n\times n}$ be regular and satisfy (3.4). We define

$$U\mathcal{E}_{s,\sigma_1}V =: \begin{pmatrix} \widehat{E}_{11} & \widehat{E}_{12} \\ \widehat{E}_{21} & \widehat{E}_{22} \end{pmatrix}, \qquad U\mathcal{B}_{s,\sigma_1} =: \begin{pmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{pmatrix}, \qquad \mathcal{C}_{s,\sigma_1}V =: \begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix}.$$

For all $s \in \mathbb{C}_{\geq 0}$ the matrices $(s\widehat{E}_{11} - \widehat{A}_{11})$ and $(s\widetilde{E}_{11} - \widetilde{A}_{11})$ are regular. Thus we have

$$\begin{pmatrix} s\widehat{E}_{11} - \widehat{A}_{11} & s\widehat{E}_{12} - \widehat{A}_{12} \\ s\widehat{E}_{21} & s\widehat{E}_{22} \end{pmatrix} [\mathbb{R}^n] = \left(UE^\top T^{-1} \begin{pmatrix} s\widetilde{E}_{11} - \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} TV \right) [\mathbb{R}^n]$$

$$\stackrel{(*)}{=} \left(UE^\top T^{-1} \right) \left[\mathbb{R}^{n-d} \times \{0\}^d \right] = \mathbb{R}^{n-d} \times \{0\}^d. \quad (3.5)$$

We obtain the last equality because $(*)$ holds for all $s \in \mathbb{C}_{\geq 0}$ and in particular for $s = 0$. That implies $\widehat{E}_{21} = 0$, $\widehat{E}_{22} = 0$ and finally $\sigma(\widehat{E}_{11}, \widehat{A}_{11}) = \sigma(\widetilde{E}_{11}, \widetilde{A}_{11})$ which means $S_2 \in \mathbb{S}^+_{n-d,p,m}$. Furthermore, we have $\widehat{B}_2 = 0$ because

$$\begin{pmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{pmatrix} [\mathbb{R}^m] = \left(UE^\top T^{-1} \begin{pmatrix} \widetilde{B}_1 \\ 0 \end{pmatrix} \right) [\mathbb{R}^m] \stackrel{(3.5)}{=} \mathbb{R}^{n-d} \times \{0\}^d.$$

Finally, we show that $\mathcal{G}(S_2) = \mathcal{G}(S_1)$. Let $s \in \rho(\widehat{E}_{11}, \widehat{A}_{11})$ and $Y_1 := \widetilde{C}_1(s\widetilde{E}_{11} - \widetilde{A}_{11})^{-1}$. We conclude with $Y := \begin{pmatrix} Y_1 & 0 \end{pmatrix} \in \mathbb{R}^{p\times n}$ and $X := \begin{pmatrix} X_1 & X_2 \end{pmatrix} := YTE^{-\top}U^{-1}$ that

$$\begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{C}_1 & 0 \end{pmatrix} TV = Y \begin{pmatrix} s\widetilde{E}_{11} - \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} TV = X \begin{pmatrix} s\widehat{E}_{11} - \widehat{A}_{11} & s\widehat{E}_{12} - \widehat{A}_{12} \\ 0 & 0 \end{pmatrix}.$$

In particular $X_1 = \widehat{C}_1(s\widehat{E}_{11} - \widehat{A}_{11})^{-1}$. That implies

$$\mathcal{G}(S_1)(s) = \widetilde{C}_1(s\widetilde{E}_{11} - \widetilde{A}_{11})^{-1}\widetilde{B}_1 = Y_1\widetilde{B}_1 = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} \begin{pmatrix} \widetilde{B}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix} TE^{-\top}U^{-1} \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix} = X_1\widehat{B}_1 = \widehat{C}_1(s\widehat{E}_{11} - \widehat{A}_{11})^{-1}\widehat{B}_1 = \mathcal{G}(S_2)(s).$$

By Theorem 3.13 we get $\|\mathcal{G}(S) - \mathcal{G}(S_2)\|_\infty = \sigma_1$. $\qquad\square$

Under the additional condition $E = I$ we can use the less complex Schur decomposition to attain (3.4) in Theorem 3.14.

Lemma 3.15. *In the situation of [Theorem 3.14](#), where $E = I$, we obtain regular matrices U and $V := U^\top$, such that [(3.4)](#) holds, by applying the Schur decomposition to $\mathcal{A}_{S,\sigma_1}$.*

Proof. Let $T \in \mathbb{R}^{n \times n}$ be as in [Lemma 3.11](#). Then we obtain

$$\left(T\mathcal{E}_{S,\sigma_1} T^{-1}, T\mathcal{A}_{S,\sigma_1} T^{-1} \right) = \left(\begin{pmatrix} \widetilde{E}_{11} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} \right)$$

with $\widetilde{A}_{11} \in \mathbb{R}^{(n-d) \times (n-d)}$ and $\sigma(\widetilde{E}_{11}, \widetilde{A}_{11}) \subseteq \mathbb{C}_{<0} \cup \{\infty\}$. In particular, $\widetilde{A}_{11}$ is regular. The matrices $\mathcal{A}_{S,\sigma_1}$ and $T\mathcal{A}_{S,\sigma_1} T^{-1}$ have the same eigenvalues counting multiplicities. By the Schur decomposition there exists a regular (in particular an orthogonal) $U \in \mathbb{R}^{n \times n}$ such that

$$U\mathcal{A}_{S,\sigma_1} U^{-1} = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{pmatrix} = UT^{-1} \begin{pmatrix} \widetilde{A}_{11} & 0 \\ 0 & 0 \end{pmatrix} TU^{-1},$$

where $\widehat{A}_{11} \in \mathbb{R}^{(n-d) \times (n-d)}$ is regular and $\sigma(\widehat{A}_{22}) = \{0\}$. This implies $\operatorname{rank}(\widehat{A}_{11}) = \operatorname{rank}(\widetilde{A}_{11}) = \operatorname{rank}(U\mathcal{A}_{S,\sigma_1} U^{-1})$ and thus $\widehat{A}_{22} = 0$. $\qquad\square$

In summary, we obtain the following result for [(AP$_\infty$)](#).

Theorem 3.16. *Let $S \in \mathbb{S}_{n,p,m}^0$. Then there exist $S_+ \in \mathbb{S}_{n_+,p,m}^+$ and $S_- = (E_-, A_-, B_-, C_-, 0) \in \mathbb{S}_{n_-,p,m}^-$ such that $S \sim S_+ \oplus S_-$. We set $\sigma_1 := \sqrt{\max \sigma(E_-^\top \mathfrak{D}_{S_-} E_- \mathfrak{C}_{S_-})}$, then*

$$\inf_{\widehat{S} \in \bigcup_{\widehat{n} \in \mathbb{N}} \mathbb{S}_{\widehat{n},p,m}^+} \|\mathcal{G}(S) - \mathcal{G}(\widehat{S})\|_\infty = \sigma_1.$$

We define the matrices

$$\mathcal{R}_{S_-,\sigma_1} := \mathfrak{D}_{S_-} E_- \mathfrak{C}_{S_-} E_-^\top - \sigma_1^2 I, \quad \mathcal{E}_{S_-,\sigma_1} := E_-^\top \mathcal{R}_{S,\sigma_1}, \quad \mathcal{B}_{S_-,\sigma_1} := E_-^\top \mathfrak{D}_{S_-} B_-,$$

$$\mathcal{A}_{S_-,\sigma_1} := -A_-^\top \mathcal{R}_{S,\sigma_1} - C_-^\top \mathcal{C}_{S_-,\sigma_1}, \quad \mathcal{C}_{S_-,\sigma_1} := C_- \mathfrak{C}_{S_-} E_-^\top.$$

If $(\mathcal{E}_{S_-,\sigma_1}, \mathcal{A}_{S_-,\sigma_1})$ is regular, then $S_+ \oplus (\mathcal{E}_{S_-,\sigma_1}, \mathcal{A}_{S_-,\sigma_1}, \mathcal{B}_{S_-,\sigma_1}, \mathcal{C}_{S_-,\sigma_1}, 0)$ solves [(AP$_\infty$)](#).
If $(\mathcal{E}_{S_-,\sigma_1}, \mathcal{A}_{S_-,\sigma_1})$ is singular, then there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ with

$$U\mathcal{A}_{S_-,\sigma_1} V = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & 0 \end{pmatrix} \text{ and regular } \widehat{A}_{11}. \tag{3.6}$$

Every regular $U, V \in \mathbb{R}^{n \times n}$ satisfying [(3.6)](#), fulfil

$$U\mathcal{E}_{S_-,\sigma_1} V = \begin{pmatrix} \widehat{E}_{11} & \widehat{E}_{12} \\ 0 & 0 \end{pmatrix}, \quad U\mathcal{B}_{S_-,\sigma_1} = \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix}, \quad \mathcal{C}_{S_-,\sigma_1} V = \begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix}.$$

The system $S_+ \oplus (\widehat{E}_{11}, \widehat{A}_{11}, \widehat{B}_1, \widehat{C}_1, 0)$ solves [(AP$_\infty$)](#).

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4 Application

Algorithm

We rewrite our results in an algorithmic manner. Note that the solution of (AP_2) is necessary to compute the solution of (AP_∞) and thus no extra computation is needed.

Input: $S \in \mathbb{S}^0_{n,p,m}$

Output: $S_q \in \mathbb{S}^+_{n_q,p,m}$ which solves (AP_q) for $q \in \{2, \infty\}$.

1. Decompose S into $S_- \in \mathbb{S}^-_{n_-,p,m}$ and $S_+ \in \mathbb{S}^+_{n_+,p,m}$ as in Theorem 3.1 with the help of Theorem 2.3.

2. The system $S_2 := S_+$ solves (AP_2).

3. Determine $\widetilde{\mathfrak{C}} := \mathfrak{C}_{s_-} E_-^\top$, $\widetilde{\mathfrak{D}} := E_-^\top \mathfrak{D}_{s_-}$ and $\sigma_1 = \sqrt{\max \sigma(\widetilde{\mathfrak{D}}^\top \widetilde{\mathfrak{C}})}$.

4. Compute $R := \widetilde{\mathfrak{D}}^\top \widetilde{\mathfrak{C}} - \sigma_1^2 I$, $\quad \widehat{E} := E_-^\top R$, $\qquad\qquad \widehat{B} := \widetilde{\mathfrak{D}} B_-$,
$$\widehat{A} := -A_-^\top R - C_-^\top \widehat{C}, \quad \widehat{C} := C_- \widetilde{\mathfrak{C}}.$$

5. If $(\widehat{E}, \widehat{A})$ is regular (test with Corollary 3.8), then (AP_∞) is solved by $S_\infty := S_+ \oplus (\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C}, 0)$.

6. If $(\widehat{E}, \widehat{A})$ is singular, then determine $U, V \in \mathbb{R}^{n \times n}$ (SVD) such that

$$U \widehat{A} V = \begin{pmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & 0 \end{pmatrix} \text{ with regular } \widehat{A}_{11}.$$

Compute $\widehat{E}_{11}, \widehat{B}_1, \widehat{C}_1$ of $U \widehat{E} V = \begin{pmatrix} \widehat{E}_{11} & \widehat{E}_{12} \\ 0 & 0 \end{pmatrix}$, $U \widehat{B} = \begin{pmatrix} \widehat{B}_1 \\ 0 \end{pmatrix}$, $\widehat{C} V = \begin{pmatrix} \widehat{C}_1 & \widehat{C}_2 \end{pmatrix}$, then
(AP_∞) is solved by $S_\infty := S_+ \oplus (\widehat{E}_{11}, \widehat{A}_{11}, \widehat{B}_1, \widehat{C}_1, 0)$.

For the numerical issues of step 1 we refer to [KD92] and of step 3 to [Pen98].

Both approximation techniques does not increase the order of the system, i.e. $n_q \le n$. For optimal RH_2-approximation we even have $n_2 < n$. Another advantage is the less computational complexity compared to RH_∞-approximation. The behaviour of the transfer functions $\mathcal{G}(S)$ and $\mathcal{G}(S_2)$ at infinity is the same, i.e. $\mathcal{G}(S)(\infty) = \mathcal{G}(S_2)(\infty)$. This does not hold for optimal RH_∞-approximation in general. However, the suboptimal approximation $S_+ \oplus \mathcal{S}_{s_-,\gamma}$ (choosing $\gamma > \sigma_1$ in Theorem 3.7) has this property. Since $\mathcal{E}_{s_-,\gamma}$ is regular, we have

$$\mathcal{G}(S)(\infty) = \mathcal{G}(S_+)(\infty) = \mathcal{G}(S_+)(\infty) + \mathcal{G}(\mathcal{S}_{s_-,\gamma})(\infty) = \mathcal{G}(S_+ \oplus \mathcal{S}_{s_-,\gamma})(\infty).$$

Examples

The following benchmarks are performed in MATLAB®. We test the building model (Figure 1) and the clamped beam model (Figure 2), see [CD02] for details. Both models are described by stable standard systems with one input and one output, i.e. $m = p = 1$.

We apply (shifted) Arnoldi method to reduce the systems. The parameters (interpolation point $s \in \mathbb{C} \cup \{\infty\}$ and reduced order $k \in \mathbb{N}$) are based on the examples in [ASG01]. The computed reduced systems are unstable as listed in Table 1. Thus we apply the algorithm presented above. Since the transfer functions of the original systems (and hence of the reduced systems) vanish at infinity an optimal RH_∞-approximation would result in huge relative errors at high frequencies. That is why we compute suboptimal approximations (i. e. $\gamma > \sigma_1$) to match the original transfer functions at infinity.

model	order	model reduction method	reduced order	unstable poles	max. real part of unstable poles
building	48	Arnoldi, $s = \infty$	31	2	≈ 42.4
		RH_2	29	-	-
		RH_∞, $\gamma = 1.001 \cdot \sigma_1$	31	-	-
clamped beam	348	Arnoldi, $s = 0.1$	13	1	≈ 1.5
		RH_2	12	-	-
		RH_∞, $\gamma = 1.001 \cdot \sigma_1$	13	-	-

Table 1: Summary of the results

The error between the original systems and the reduced systems of Arnoldi method does not significantly chance after applying our stabilization algorithm as depicted in Figure 1 and Figure 2.
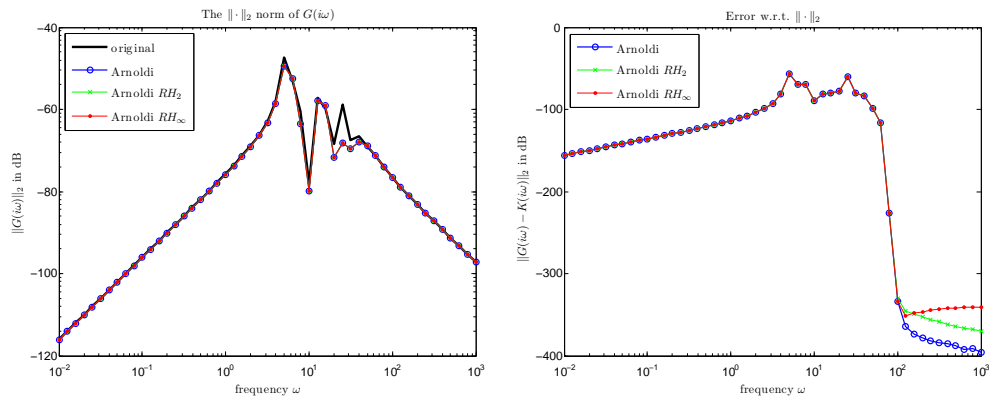

Figure 1: Frequency responses of the (reduced) building model (left) and the error systems (right)

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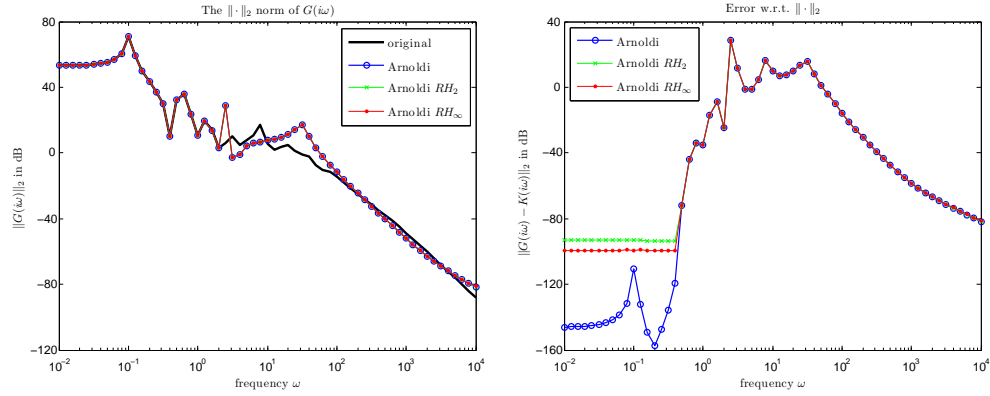
Figure 2: Frequency responses of the (reduced) clamped beam model (left) and the error systems (right)

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